1 Talk Summary

This talk mostly follows the structure of Chapter 44 of the stacks project [Sta22, Tag 0B94]. In this talk we aim to introduce the concept of a Hilbert scheme of points and give some results on the representability of Hilbert schemes. The motivation being that we just love to study things that are named for Hilbert. (Or, alternatively, because we aim to study Jacobian Varieties which represent the Picard functor which is related to the Hilbert functor via the Abel- Jacobi map).

2 Hilbert scheme of points

2.1 Introducing the Hilbert functor

We start by introducing the functor $\mathrm{Hilb}^d_{X/S}$ for a morphism of schemes $X \to S$ which we can then show that, with some conditions on $X \to S$, is representable by a scheme which defines our 'Hilbert Scheme'.
Before defining the Hilbert functor it would be useful to recall what it means for a morphism of schemes to be finite locally free.

Definition 1. For $f : X \to S$ a morphism of schemes, we call f *finite locally free* if f is affine an $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module. The *rank* of the morphism refers to the rank of $f_*\mathcal{O}_X$ as an $\mathcal{O}_S$-module.

Remark. A useful fact we may use is that f is finite locally free if and only if it is finite, flat and locally of finite presentation. [Sta22, Tag 02KB] but in fact in general this second of the "morphism of schemes" portion of the stacks project is useful to understand some proofs...

Definition 2. (Hilbert functor)

- We fix a base S. Let $X \to S$ be a morphism of schemes and $d \geq 0$ an integer. For a scheme T over S we define
$$\mathrm{Hilb}^d_{X/S}(T) = \left\{ \begin{smallmatrix} Z \subset X_T \text{ closed subscheme such that} \\ Z \to T \text{ is finite locally free degree } d \end{smallmatrix} \right\}.$$
- If $T' \to T$ is a morphism of schemes over S and if $Z \in \mathrm{Hilb}^d_{X/S}(T)$, then the base change to T', $Z_{T'} \subset X_{T'}$ is an element of $\mathrm{Hilb}^d_{X/S}(T')$ (recall from AG1 for example that being a closed subscheme, finite locally free are stable under base change). Hence $\mathrm{Hilb}^d_{X/S}$ is a functor
$$\mathrm{Hilb}^d_{X/S} : (\mathbf{Sch}/S)^{\mathrm{opp}} \to \mathbf{Sets}.$$

During this talk we will see that for $X \to S$ such that any finite set of points in a fibre are contained in an open affine then $\mathrm{Hilb}^d_{X/S}$ is representable by a scheme. We will denote this scheme $\underline{\mathrm{Hilb}}^d_{X/S}$!

Remark. Apparently, for any morphism of schemes $X \to S$ we know that $\mathrm{Hilb}^d_{X/S}$ is actually an algebraic space! An algebraic space is a nice generalisation of a scheme. Roughly, an algebraic space is to the etale topology what schemes are to the Zariski topology but understanding this is a massive tangent and super unnecessary for the scope of this seminar as the condition required for this functor to be representable by a scheme will encompass the situation we are interested in (but it's a titbit I found interesting and I'm the speaker so Ha!). For those who are interested this result is actually mentioned in chapter 98(!) of the stacks project (but this is a rabbit hole that I didn't go all the way down).

2.2 Representability

Before talking representability we might want to recall a few notions from in and around the Yoneda lemma that will be useful for the following proofs

- **What does it mean to be representable?** Recall that a functor $F : \mathcal{C} \to \mathbf{Set}$ is representable by the object U if and only if there exists an object $U \in \mathcal{C}$ such that $h_U = F$ where $h_U = \text{Mor}(-, U) : \mathcal{C} \to \mathbf{Sets}$.
- **Yoneda Lemma** For $U, V \in \text{Ob}(\mathcal{C})$, for any transformation $s : h_U \to h_V$ there is a unique morphism $\varphi : U \to V$ such that $h(\varphi) = s$. I.e. there is a natural bij

$$\begin{aligned} \text{Mor}(h_U, F) &\to F(U) \\ s &\mapsto s_U(\text{id}) \end{aligned}$$

  for any contravariant functor F.
- **Universal objects** The Yoneda lemma says that the choice of object representing F, if it exists is unique, as is the choice of isomorphism $s : h_U \to F$. Also by the Yoneda lemma we have some object $e \in F(U)$ which we call the universal object, corresponding to this natural transformation s. It has the property that for any $V \in \mathcal{C}$, $\text{Mor}(V, U) \to F(V), f \mapsto F(f)(e)$ is a bijection.

Now we introduce some elementary results leading up to the desired result on representability of $\text{Hilb}^d_{X/S}$. For S a scheme and $i : X \to Y$ a closed immersion of S-schemes. Then there is a transformation of functors

$$\text{Hilb}^d_{X/S} \to \text{Hilb}^d_{Y/S}$$

such that for a scheme T over S,

$$\begin{aligned} \text{Hilb}^d_{X/S}(T) &\to \text{Hilb}^d_{Y/S}(T) \\ Z &\mapsto i_T(Z_T) \end{aligned}$$

where $i_T : X_T \to Y_T$ is the base change of $i : X \to Y$ to T.

Lemma 1. *Let S be a scheme. Let $i : X \to Y$ be a closed immersion of schemes. If $Hilb^d_{Y/S}$ is representable by a scheme, so is $Hilb^d_{X/S}$ and the corresponding morphism of schemes $\underline{Hilb}^d_{X/S} \to \underline{Hilb}^d_{Y/S}$ is a closed immersion.*

Proof.
- Let T be a scheme over S and let $Z \in \text{Hilb}^d_{Y/S}(T)$.
- **Claim:** There is a closed subscheme $T_X \subset T$ such that the morphism of schemes $T' \to T$ factors through T_X if and only if $Z_{T'} \to Y_{T'}$ factors through $X_{T'}$.
- Why is this claim enough? Let T_{univ} be the scheme representing $\text{Hilb}^d_{Y/S}$ and $Z_{\text{univ}} \in \text{Hilb}^d_{Y/S}(T_{\text{univ}})$ universal object (i.e. the unique element of $\text{Hilb}^d_{Y/S}(T_{univ})$ corresponding to the Yoneda transformation $h_{T_{\text{univ}}} \to \text{Hilb}^d_{Y/S}$). Applying our claim here we have to this situation we have a closed subscheme $T_{\text{univ},X} \subset T_{\text{univ}}$ such that $Z_{\text{univ},X} = Z_{\text{univ}} \times_{T_{\text{univ}}} T_{\text{univ},X}$ is a closed subscheme of $X \times_S T_{\text{univ},X}$ and hence defines an element of $\text{Hilb}^d_{X/S}(T_{\text{univ},X})$. By some formal argument involving some unpacking of defintions and base change we can show that then $T_{\text{univ},X}$ represents $\text{Hilb}^d_{X/S}$ with representing object $Z_{\text{univ},X} \in \text{Hilb}^d_{X/S}(T_{\text{univ},X})$.
- *Proof.* (Of claim) We give a sketch that includes some reduction of cases without justification.
  - Let $Z' = X_T \times_{Y_T} Z$. Given $T' \to T$ we see that $Z_{T'} \to Y_{T'}$ factors through $X_{T'}$ if and only if $Z'_{T'} \to Z_{T'}$ is an isomorphism. Hence we now aim to find T_X such that $T' \to T$ factors through T_X if and only if $Z'_{T'} \to Z_{T'}$ is an isomorphism.
  - It is possible to reduce to the case where $T = \text{Spec}A$, $Z = \text{Spec}B$.
  - By definition $Z \in \text{Hilb}^d_{Y/S}(T)$ means that Z is a closed subscheme of Y_T with $Z \to T$ finite, locally free degree d.

– Hence we may shrink T further and assume there is an isomorhism $\varphi : B \xrightarrow{\sim} A^{\oplus d}$ as A-modules.
– The $Z' = \mathrm{Spec}(B/J)$ for some ideal $J \subset B$. (Because X is some closed subscheme of Y so its cut out by some sheaf of ideals, look at this on $Z = \mathrm{Spec}(B)$).
– Let $g_\beta \in J$ be a collection of generators and write $\varphi(g_\beta) = (g_\beta^1, ..., g_\beta^d)$. Then T_X must be given by $\mathrm{Spec}(A/(g_\beta^j))$.

Alternatively we can use some lemma in the stacks project to prove this less directly and more succinctly but the statement of this was a bit gross and needed a lot of unpacking. For those interested the lemma in question was [Sta22, Tag 05PC]. □

□

Lemma 2. *Let $X \to S$ be a morphism of schemes. If $X \to S$ is separated and $\mathit{Hilb}^d_{X/S}$ is representable, then $\underline{\mathit{Hilb}}^d_{X/S} \to S$ is separated.*

Proof.
- Let $H = \underline{\mathrm{Hilb}}^d_{X/S}$ and let $Z \in \mathrm{Hilb}^d_{X/S}(H)$ be the universal object.
- Consider $Z_1, Z_2 \in \mathrm{Hilb}^d_{X/S}(H \times H)$ we get by pulling back the two projections $pr_1, pr_2 : H \times H \to H$. Then $Z_1 = Z \times H \subset X_{H \times H}$ and $Z_2 = H \times Z \subset X_{H \times H}$.
- Since H represents the functor $\mathrm{Hilb}^d_{X/S}$, the diagonal morphism $\Delta : H \to H \times H$ has the following universal property: A morphism of schemes $T \to H \times H$ factors through Δ is and only if $Z_{1,T} = Z_{2,T}$ as elements of $\mathrm{Hilb}^d_{X/S}(T)$.
- Let $Z = Z_1 \times_{X_{H \times H}} Z_2$. Then we see that $T \to H \times H$ factors through Δ if and only if the morphisms $Z_T \to Z_{1,T}$ and $Z_T \to Z_{2,T}$ are isomorphisms.
- We use the claim in the proof of Lemma 1 to conclude that Δ is in fact a closed immersion.

□

Lemma 3. *Let $X \to S$ be a morphism of affine schemes. Let $d \geq 0$. Then $\mathit{Hilb}^d_{X/S}$ is represntable.*

In order to prove the lemma this we will use the following result on criteria for representability without proof (for the proof see [Sta22, Tag 01JJ]):

Lemma 4. *Let F be a functor $F : (\mathbf{Sch}/S)^{opp} \to \mathbf{Sets}$. Suppose that*

1. *F satisfies the sheaf property for the Zariski topology*
2. *There is a set I and a collection of subfunctors $F_i \subset F$ such that:*
   - *each F_i is representable*
   - *each $F_i \subset F$ is representable by an open immersion*
   - *the F_i cover F*

then F is representable.

However, this needs some unpacking. Some questions we might have to understand this lemma (for more details on this read the topologies of schemes entry on stacks project [Sta22, 020K]):

- **What do we mean for a functor out of schemes to satisfy the sheaf condition?**
  – **Topology on the category of schemes** In general to define the sheaf property on a category we use something called a *Grothendieck topology* on the category, by analogue of the category of opens of a topological, by specifying families with certain properties which we will call *coverings*. For more details on this the section on representability criterion in the stacks project [Sta22, Tag 01JF]. For us, the important thing for us is that we can define such a topology on $\mathbf{Sch}/S$ called the Zariski topology. A cover in the Zariski topology on $\mathbf{Sch}/S$ is a collection of morphisms $\{T_i \to T\}_I$ which are all *open immersions* such that the image of the T_i cover T. In particular, covers in the Zariski topology are just the covers usual notion of covers of each scheme.

- **Sheafs from the category of schemes** We define the sheaf condition on the category $\mathbf{Sch}/S$ with some topology by saying that for any $\{T_i \to T\}_i$ a cover the following is an equaliser

$$F(T) \to \prod_{i \in I} F(T_i) \rightrightarrows \prod_{i,j \in I} F(T_i \times_T T_j)$$

(does this look familiar?). In particular this means that for Hilb to be a sheaf for the Zariski topology it is equivalent to be objejctwise a sheaf.

- Note that we can make sense of the same idea of (for example) the *fpqc topology* where, instead of requiring the morphisms to be open immersions, we instead require them to be fpqc morphisms (similarly for étale topology etc). The fpqc topology is finer than the Zariski topology (any open immersion is fpqc) and it is actually possible to prove that in fact $\mathrm{Hilb}^d_{X/S}$ is a sheaf for the fpqc topology, an even stronger statement than $\mathrm{Hilb}^d_{X/S}$ being a sheaf for the Zariski topology [Sta22, Tag 0B95].

- **What do we mean by F_i covering F?**

  Definition 3. We say a collection of subfunctors $F_i \subset F$ *cover* F if for every $e \in F(T)$ there exists an open covering $T = \bigcup T_i$ such that $e|_{T_i} \in F_i(T_i)$.

- **What do we mean to say that a subfunctor is representable by an open immersion?**

  Definition 4. We say $F' \subset F$ is *representable by an open immersion* if for all pairs (T, e), T a scheme, $e \in F(T)$ there is an open subscheme $U_e \subset T$ such that a morphism $f : T' \to T$ factors through U_e if and only if $f^*e \in F(T')$.

  Remark. If we know that the all the subfunctors F_i are representable by open immersions, to check that these cover it suffices to check that $F(T) = \bigcup F_i(T)$ for T is the spectrum of a field.

Proof. (Of Lemma 3) We offer some sketch of this proof which does not require us to understand all the ins and outs of Grothendieck topologies or sheaves of schemes but will give some idea how we can use this lemma to show this statement.

- Let $S = \mathrm{Spec} R$. We can choose a closed immersion of X into the spectrum of $R[x_i]_{i \in I}$ for some I of sufficiently large cardinality. Hence by lemma 1 we can assume $X = \mathrm{Spec}(A)$ with $A = R[x_i]_{i \in I}$.

- As mentioned above, $\mathrm{Hilb}^d_{X/S}$ is even a sheaf for the fpqc topology, an even stronger condition. Hence (1) is satisfied. For a sketch of the proof we use the fact that a cover under base change gives a cover of the base change and then apply descent results to see that the relevant thing is in fact an equaliser.

- In order to see (2) we will specify a cover by representable subfunctors: For every $W \subset A$ of cardinality d we construct a subfunctor F_W of $\mathrm{Hilb}^d_{X/S}$.

- Observe that for any $Z \in \mathrm{Hilb}^d_{X/S}(T)$ we have an injective $\mathcal{O}_T$-linear map

$$\bigoplus_{f \in W} \mathcal{O}_T \to (Z \to T)_* \mathcal{O}_Z \qquad (*)$$
$$(g_f) \mapsto \sum g_f f|_Z$$

  where for $f \in A, Z \in \mathrm{Hilb}^d_{X/S}(T)$ we write $f|_Z$ to mean the pullback of f by the morphism $Z \to X_T \to X$. We let

$$F_W(T) = \left\{ Z \in \mathrm{Hilb}^d_{X/S}(T) \text{ such that } * \text{ is surjective} \right\}.$$

- We now need to check that these are open, representable and cover F.

- **Open** This is a consequence of the result in commutative algebra that says that for a morphism of finite projective R-modules $\varphi : P_1 \to P_2$, the set of primes $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak{p})$ is surjective is open and for any $f \in R$ such that $D(f) \subset W$ we have $P_{1,f} \to P_{2,f}$ is surjective (this is a purely commutative algebra result and is point (2) of [Sta22, Tag 00O0]).
- **Cover** Since
$$A \otimes_R \mathcal{O}_T = (X_T \to T)_* \mathcal{O}_{X_T} \to (Z \to T)_* \mathcal{O}_Z$$
is surjective and $(Z \to T)_* \mathcal{O}_Z$ is finite locally free of rank d, for every point $t \in T$ we can find a finite subset $W \subset A$ of cardinality d, whose image forms a basis of the d-dimensional $\kappa(t)$-vector space $((Z \to T)_* \mathcal{O}_Z)_t \otimes_{\mathcal{O}_{T,t}} \kappa(t)$. By Nakayama lemma there is an open neighbourhood $V \subset T$ of t such that $Z_V \in F_W(V)$.
- We can show these subfunctors are representable by an affine scheme. I will describe the construction but its going to sound like a lot of symbols and as a proof this serves better being read and pondered.
- Let $W \subset A$ be a subset of cardinality d.
- Call the elements of W, $f_1, ..., f_d$. We aim to construct a universal element $Z_{\mathrm{univ}} = \mathrm{Spec}(B_{\mathrm{univ}})$ of F_W over the representing scheme $T_{\mathrm{univ}} = \mathrm{Spec}(R_{\mathrm{univ}})$ where
$$B_{\mathrm{univ}} = R_{\mathrm{univ}}[\bar{f}_1, ..., \bar{f}_d]/(\bar{f}_k \bar{f}_l - \sum c_{kl}^m \bar{f}_m)$$
where $\bar{f}_l$ will be the images of the f_l. The closed immersion $Z_{\mathrm{univ}} \to X_{T_{\mathrm{univ}}}$ is given by the ring map
$$A \otimes_R R_{\mathrm{univ}} \to B_{\mathrm{univ}}$$
mapping $1 \otimes 1$ to $\sum b^l \bar{f}_l$ and x_i to $\sum b_i^l \bar{f}_l$.
- We in fact claim that F_W is represented by the spectrum of the ring $R_{\mathrm{univ}} = R[c_{kl}^m, b^l, b^i]/\mathfrak{a}_{\mathrm{univ}}$. How do we define $\mathfrak{a}_{\mathrm{univ}}$? First consider the ideal $\mathfrak{a}'_{\mathrm{univ}}$ such that:
  * multiplication on B_{univ} is commutative i.e. $c_{lk}^m - c_{kl}^m \in \mathfrak{a}'_{\mathrm{univ}}$,
  * multiplication on B_{univ} is associative i.e. $c_{lk}^m c_{mn}^p - c_{lq}^p c_{kn}^q \in \mathfrak{a}'_{\mathrm{univ}}$,
  * $\sum b^l \bar{f}_l$ is multiplicative 1 in B_{univ} i.e. we should have that $(\sum b^l e_l) e_k = e_k$ for all k. Explicitly, $\sum b^l c_{lk}^m - \delta_{km} \in \mathfrak{a}'_{\mathrm{univ}}$.

  After dividing out by the ideal generated by the above elements we have a ring map
$$\Psi : A \otimes_R R[c_{kl}^m, b^l, b_i^l]/\mathfrak{a}'_{\mathrm{univ}} \to (R[c_{kl}^m, b^l, b_i^l]/\mathfrak{a}'_{\mathrm{univ}})[\bar{f}_1, ..., \bar{f}_d]/(\bar{f}_k \bar{f}_l - \sum c_{kl}^m \bar{f}_m)$$
  sending $1 \otimes 1$ to $\sum b^l \bar{f}_l$ and $x_i \otimes 1$ to $\sum b_i^l \bar{f}_l$. Hence we need to also add elements to our ideal which require f_l to map to $\bar{f}_l$ in B_{univ}. We write $\Psi(f_l) - \bar{f}_l = \sum h_l^m \bar{f}_l$ with $h_l^m \in R[c^m, b^l, b_i^l]/\mathfrak{a}'_{\mathrm{univ}}$ then we need h_l^m to be 0. Hence is the ideal
$$\mathfrak{a}_{\mathrm{univ}} = \mathfrak{a}'_{\mathrm{univ}} + (\text{lifts of } h_l^m \text{ to } R[c_{kl}^m, b^l, b_i^l])$$
  then we can see that by construction F_W is represented by $\mathrm{Spec}(R_{\mathrm{univ}})$.

□

Proposition 5. *Let $X \to S$ be a morphism of schemes and $d \geq 0$. Assume for all $(s, x_1, ..., x_d)$ where $s \in S$ and $x_1, ..., x_d \in X_s$ there exists an affine open $U \subset X$ with $x_1, ..., x_d \in U$. Then $\mathrm{Hilb}_{X/S}^d$ is representable.*

Proof.
- We can reduce to the case where S is affine by some gluing argument.
- For $U \subset X$ affine open, denote $F_U \subset \mathrm{Hilb}_{X/S}^d$ the subfuctor such that for a scheme T/S an element $Z \in \mathrm{Hilb}_{X/S}^d(T)$ is in $F_U(T)$ if and only if $Z \subset U_T$.
- Again we can use lemma 4 to prove this statement.
- Again (1) is satisfied as before.

- Now, these subfunctors are representable by the last lemma and the fact that they cover comes from our assumption as follows:
  We want to see that if T is the spectrum of a field and $Z \subset X_T$ is a closed subscheme finite, flat of degree d over T then $Z \to X_T \to X$ factors through an affine open U of X. From our assumption on X/S, Z will have at most d points which will map into the fibre of X over the image point of $T \to S$.

$\square$

Remark. The condition that any finite set of points in any fibre are contained in an affine open isn't that tangible but actually a lot of classes of morphisms we would like Hilb to be representable for have this property. Some examples of $f : X \to S$ with this property (and hence for which the conclusion of proposition 5 holds):

1. X is quasi-affine,
2. f is quasi- affine
3. f is quasi-projective,
4. f is locally projective,
5. There exists an ample invertible sheaf on X,

And also, for those who are familiar with the concept of f-ample and f-very ample sheaves the following

6. there exists an f-ample invertible sheaf on X
7. There exists an f-very ample invertible sheaf on X.

In each of these cases we can prove the property by [Sta22, Tag 01ZY]. (Which says that if a scheme is quasi affine, locally isomorphisc to a locally closed subscheme of an affine scheme, has an ample invertible sheaf or the scheme X is isomorphic to a locally closed subscheme of $\mathrm{Proj}(S)$ of a graded ring S then any finite subset is contained in an open affine). In particular for this seminar we will care about applying our theory to smooth projective curves hence this property is enough for $\mathrm{Hilb}^d_{X/S}$ to be representable in the case where X is a curve over S.

References

[Sta22] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2022.